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Determining Work Standards From Data With a Weak Dependent Variable Using the Beta Distribution.

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Determining Work Standards From Data With a Weak Dependent Variable Using the Beta Distribution

This paper deals with the problem of statistically determining work standards from production data with a weak dependent variable. The weakness of the dependent variable lies in the fact that it represents available man hours rather than applied man hours and is therefore purely a boundary condition. Classical methods, such as linear programming and simple multiple regression, do not take this problem into account and therefore give solutions which are untrustworthy.

In this procedure random variables, which approximate order statistics, drawn from the beta distribution are used to modify the dependent variable. These random variables represent the percent of time each day which was actually applied to units of work. The beta distribution is used because it is confined to the unit interval and is very flexible in form. Dealt with are the simultaneous problems of finding progressively better orderings of the data from slackest to busiest and determining the optimal form of the beta distribution as defined by its two parameters, alpha and beta. The criterion used to define optimality is the F statistic from the analysis of variance on the multiple regression. The ordering problem was handled by performing interchanges on the days which interchanges improved the F statistic. The optimal form of the beta distribution was determined by the two stage process of locating the ridge of the three dimensional surface and then locating the peak of that ridge.

This procedure shows some very favorable results. The residuals are reduced from those of the simple multiple regression by a factor of twenty. The final results do not violate the boundary condition of the dependent variable which the simple multiple regression did. The results are given extra credibility by the fact that Mondays were clearly among the slackest days, a phenomenon which is generally accepted as being true.

Further experimentation with the data showed that either some of the assumptions of the procedure might be questionable or the F statistic may not be the most stable criterion for optimality. These are areas which should be explored further. Also, it was not proved by this paper that the optimal order had been achieved. Since there exist twenty-five factorial different orderings, it would be impractical to test each one for ultimate validation.

SECTION 1

INTRODUCTION

Obtaining valid, legitimate work standards has been an industrial problem for many decades. Work standards represent the usual amount of time it takes to perform defined units of work, and such data can be of paramount importance to a corporation of any size.

The value of accurate work standards is multifold. In many cases, the most important gain, although also the most unquantifiable and intrinsic, is that management is given clearer understanding of what the operation entails; in a sense, a more "hands on" feel for the production process, i.e. what is capable of being done and what is not. This alleviates feelings of frustration and distance. On a more practical level, there are other bonuses. Training program usefulness can be evaluated, different modes of operation and plant layout can be compared by productivity, and forecasting of work force requirements in cyclical businesses can lessen over- and under-staffing problems. Scheduling of orders can be improved, and so the reputation of the company.

If a quick means of obtaining work standards were available, a company might also be more apt to take a risk in implementing a new mode of operation which the employees would prefer. If the production measure proved positive, management would gain in profits and employees in higher job satisfaction.

However, there is no simple, inexpensive, global method of obtaining work measurement standards. Time and motion study is widely and effectively used in certain situations, particularly in defining work standards for highly repetitive and standardized tasks. But there are several drawbacks to this method. In the first place, it is costly and time consuming. Trained professionals must accurately define the study, determine a period when production can be expected to be fairly typical, carry out the measurements and analyze the data. The study will in the end still only be as accurate as the task definitions, data gathering and analysis. There will always be the question as to whether the measurement period was representative, or whether the employees were intimidated into non-standard work practices by the presence of an observer. This type of study may also be influenced by the time available to perform it, and the only sure way to reduce this degree of uncertainty is to expend the extra time and money to gather further data. Lastly, it is unsuitable in a job shop situation when a unit of labor may not be like the previous unit, or for many work situations, especially when indirect labor is to be defined and measured. By indirect labor, for the purpose of this study, it is meant labor which is either a part of the corporate overhead but not directly related to production, such as clerical tasks, or labor which is directly related to production but not to a specific task, such as getting more raw materials or cleaning up work area. On a higher level, research could be included in this definition, but in this case, units of work become very difficult to define.

Statistical analysis of units of work produced in a given amount of employee time might circumvent many of these problems. Available account data would be used. Enough data would be on hand so that seasonal adjustments or other anomalies could be studied or evened out over time. It would be faster, take less professional personnel time and lessen the problem of indirect labor. Indirect labor would, and should be quantified as a part of the corporate overhead, however, there would no longer be the problem of the detailed definition necessary for time and motion study.

In the next section, some classic mathematical methods will be shown through the use of a set of production data. Finally, this paper presents an alternative way of analyzing such data.

SECTION 2

CLASSICAL METHODS

Introduction

An increasing proportion of jobs today are a part of the service sector of the economy. Examples include clerical, maintenance and decision-making functions. This increase in indirect labor has made work standards more difficult to determine, since formerly used stopwatch techniques are not always appropriate. However, data which are commonly available to businesses are the number of units of products turned out in a given time—such as a day or a week—and the total number of man hours paid for to produce those units. Such data are all that are needed for the analysis proposed by this paper.

Total man hours can simply equal the number of employees at work multiplied by the average number of hours each worked. The average number of hours each worked could be either a general labor hours figure which the company uses, or time card information. Units of products are the number of completed tasks accomplished by the employees during the time period.

These tasks might include such things as typing, filing, or maintenance.

Two mathematical methods used to analyze this type of data in the past have been linear programming and multiple regression. Following is a set of such data and the results given by the above two methods.

Data

The data given in Table 1 are from *Cost Improvement, Work Sampling, and Short Interval Scheduling* by Wallace Richardson (1976, p. 221). The independent variables, labeled X, are the numbers of different types of sales orders which were processed on flexowriters in the machine room of a sales order office. Y, the dependent variable, is the number of man hours used to accomplish these tasks. Twenty-five days worth of data were collected. Inspection of the dependent variable shows that it is obviously an imprecise measurement of applied man hours and simply is the number of employees at work on a given day multiplied by eight, with adjustment for over- or under-time. The following two methods have been used with some success to determine work standards from such data.

Linear Programming

There are several methods of formulating the linear program for the type of problem and all yield slightly different values for the coefficients. The first is:

Maximize

$$b_1+b_2+b_3+b_4+b_5+b_6$$

Subject to

$$197b_1+155b_2+211b_3+360b_4+171b_5+17b_6+W_1 = 80$$

$$187b_1+113b_2+194b_3+236b_4+196b_5+71b_6+W_2 = 80$$

$$\vdots$$

$$330b_1+116b_2+184b_3+86b_4+192b_5+128b_6+W_{25} = 94$$

The W's, slack variables, represent idle employee time and serve the same purpose as changing the equality of each constraint to less-than-or-equal-to. The solutions for the coefficients are:

$b_1 = 0.0$	$b_4 = 0.0$
$b_2 = 0.399$	$b_5 = 0.0$
$b_3 = 0.0$	$b_6 = 0.373$

The mean slack is 27.2 hours per day. As can be seen, this solution is not intuitively practical since the implication is that units 1, 3, 4, and 5 can be produced in zero time.

The second formulation is similar to the first.

Minimize

$$W_1+W_2+W_3+W_4+W_5+\ldots+W_{25}$$

Subject to

$$197b_1+155b_2+211b_3+360b_4+171b_5+17b_6+W_1 = 80$$

$$187b_1+113b_2+194b_3+236b_4+196b_5+71b_6+W_2 = 80$$

$$\vdots$$

$$330b_1+116b_2+184b_3+86b_4+192b_5+128b_6+W_{25} = 94$$

with the solutions:

$b_1 = 0.027$ $b_4 = 0.021$

$b_2 = 0.16$ $b_5 = 0.09$

$b_3 = 0.115$ $b_6 = 0.171$

The mean slack is 5.6 hours per day. Minimizing the slack variables has in this case avoided the problems of non-basic b variables.

The third formulation, and in many cases the most appealing is:

Minimize

$$(W_1+Z_1)+(W_2+Z_2)+(W_3+Z_3)+\ldots+(W_{25}+Z_{25})$$

Subject to

$$197b_1+155b_2+211b_3+360b_4+171b_5+17b_6+W_1-Z_1 = 80$$

$$187b_1+113b_2+194b_3+236b_4+196b_5+71b_6+W_2-Z_2 = 80$$

$$\vdots$$

$$330b_1+116b_2+184b_3+86b_4+192b_5+128b_6+W_{25}-Z_{25} = 94$$

The solutions are:

$b_1 = 0.029$ $b_2 = 0.114$

$b_3 = 0.158$ $b_5 = 0.057$

$b_4 = 0.027$ $b_6 = 0.223$

$\overline{W} = 2.2$ $\overline{Z} = 1.6$

This formulation allows the problem more freedom in that there is leeway given on both sides of the resource variables for plus and minus slack. However, in this situation, this leeway may not be an advantage since the resource variable may be thought of as an absolute maximum. In other words, it is extremely doubtful if a value of eighty hours has been rounded down, since overtime is usually noted.

The problem with a linear programming technique is that there is the unanswered question "How good is it?". There are no performance indicators, for the simple reason that none are needed. The validity of any linear program depends entirely on the logic of its use and the precision of the variables to which it is sensitive. The imprecision of the resource variable, therefore, is problematic for this approach.

Multiple Regression

Assuming a linear relationship, the objective of multiple regression is to determine the coefficients of the equation:

$$Y = b_0+b_1X_1+b_2X_2+b_3X_3+b_4X_4+b_5X_5+b_6X_6$$

for the data previously given in Table 1. Again, Y is the dependent variable, b_0 the average idle or unaccounted time, the other b's the time it takes to produce a unit of work, and the X's are the units of work produced.

Solving for the unknowns by minimizing the residual sum of squares we have:

$b_0 = 52.23$	$b_4 = 0.002$
$b_1 = 0.009$	$b_5 = 0.021$
$b_2 = 0.069$	$b_6 = 0.073$
$b_3 = 0.054$	

The magnitude of the unaccounted time, b_0, is unsettling. It seems to imply that on the average over two-thirds of employee time is not directly applied to work.

Unlike the linear programming method, there are performance indicators with multiple regression. The correlation coefficient for the above equation is 0.66, showing that the equation has some predictive power, but not an overwhelming amount. The percent of variance accounted for by the regression, as shown by the coefficient of determination, is 43.7. Again, this is a weak indication of a good model since less than fifty percent of the variance is explained by the equation. Another indicator is the F statistic from the analysis of variance performed on the regression. Since the F value is the ratio, the regression mean square divided by the residual mean square, it is obvious that the larger the F value, the better the model fits the data. For the above model, the F statistic is 2.33 which means that there is an eight percent chance that there is no linear relationship between the X's and Y, a hypothesis which therefore can not be rejected.

Figure 1 shows a normal probability plot of the residual from the equation. By definition, the residual of a good linear model will be normally distributed and the normal probability plot will approximate a straight line. The W test (Hahn and Shapiro, 1967, pp. 295-298), devised by Wilk, a test to evaluate the assumption of an underlying normal distribution, was run on these residuals. The test yielded a z value of -0.0658, and from a normal distribution table, $Pr(z \leq -0.0658) = 0.4721$. Since this value represents the approximate probability that the residuals are normally distributed, there is not a strong reason to accept or reject the hypothesis that the normal distribution is the underlying distribution of the residuals.

There can be other problems which arise using regression as can be seen in the study by Martin (1971) in which some of the coefficients are negative, which is obviously fallacious.

Summary

The above two mathematical approaches to the problem have been used, especially to get a quick handle on work standards. However, the impact of the weakness of the dependent variable has not been considered or compensated for.

SECTION 3

PROCEDURE AND RESULTS

Introduction

The inherent problem with the type of data in Table 1 is that the dependent variable, Y, is an imprecise measurement of man hours applied to units of work. In fact, it is not at all a measure of applied man hours but rather purely a measure of available man hours, which is a totally different thing, as it does not represent an estimate but merely an upper limit. This weakness must be compensated for before credible results can be gotten. In the previous section the equation was fitted to the data, but perhaps a better approach would be to take into account the weakness of the dependent variable, massage it, and fit the data to the equation.

It is intuitively obvious that some work days are busier than others and have less idle time, and other days are just the opposite. The amount of idle time per day varies. The average daily idle time was accounted for in the multiple regression with the constant term, b_0, and the fluctuation by the residual for each day's observation. The problem with this is that, due to the nature of least squares, approximately half of the residuals will be negative, implying that more time was applied than the amount defined by the resource variable. In other words, using the seventh day as an example, 52.23 hours were spent idle and additionally there were 10.2 hours of unrecorded labor. This situation is unlikely. In fact, it has already been stated that Y represents man hours available, and therefore is an upper limit on man hours applied.

A better way to handle idle time and its daily fluctuation would be to represent it as a distribution of percentages of the resource term, Y. Therefore, Y would be modified so that it truly represents man hours applied to the tasks.

The Beta Distribution

Since it has already been assumed that Y represents an upper limit on the hours worked, it must be that the distribution used as a multiplier of Y can only contain random variables in the range of zero to one. Also, since the shape of the distribution is unknown, the function must be flexible in form. The beta distribution is such a useful class of distributions when the random variables are restricted to the unit interval. Its density function defined over the interval (0,1) is

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha)} x^{\alpha-1}(1-x)^{\beta-1} & 0 \le x \le 1,\ 0<\alpha,\ 0<\beta \\ 0 \text{ elsewhere} & \end{cases}$$

It is related to the uniform, t, and F distributions and can be single peaked, U shaped, J shaped, or reverse J shaped. It can also be skewed or symmetrical. For these reasons it is a defensible choice to use the beta distribution to represent the percent of time actually worked.

Since twenty-five days worth of data are being analyzed, twenty-five random variables must be drawn from the beta distribution. Not being able to make any further assumptions about the data, other than that it is representative, one approach is to choose twenty-five values of p such that one twenty-sixth of the distribution lies between adjacent values and outside each of the two extreme values. In effect, these values of p represent an assumed percentage of time actually applied each day. These values of p are not really order statistics, but they do approximate the expected values of the order statistics. Routine MDBETI of the IMSL package was used to generate the twenty-five values of p from the beta distribution.

The generalized equation for the multiple regression for these data then becomes:

$$p_i Y_j = b_0 + b_1 X_{1j} + b_2 X_{2j} + b_3 X_{3j} + b_4 X_{4j} + b_5 X_{5j} + b_6 X_{6j}$$

where the p_i's are the values of the random variable p from the beta distribution. This leaves us with two obvious interdependent problems. The first is determining the alpha and beta for the beta distribution which maximizes the F statistic from the multiple regression. The second is finding the best order of the days from slackest to busiest such that the appropriate p_i is matched to the appropriate day. The following procedure will tackle these two problems.

Procedure

There are several logical starting orders for the data. Three of them can be gotten from the linear programming solutions given in the previous section, by ranking the days according to the proportion of slack to resource variable. Another option is to start with the order suggested by the multiple regression. The last option is to simply take the data in the order as originally collected.

If the number of day's worth of data is very large, the simplest approach would be to start with the data in the order as originally collected, since manual manipulation can be tedious. In this case, however, because the data set is small, it seems wise to start with a more mathematically defined order, either from the multiple regression or the linear programming solutions. They all yield different orders, so the choice is arbitrary. This procedure starts with the order from the multiple regression.

An initial alpha and beta must also be chosen. Although this choice is also arbitrary, an intuitive feel for working habits suggests some better initial values than others. A reasonable assumption is that on the average people work about 70% of every working day. Therefore

$$E(p) = \frac{\alpha}{\alpha+\beta} = 0.70$$

from which it can be derived that $\alpha = 2.33\beta$. Additionally, to allow the tail of the distribution to spread out, the standard deviation should equal a small enough value such that the expected value plus three sigma is not greater than 1.0, the upper limit of the beta distribution. A standard deviation of 0.1 satisfies this imposed constraint.

Therefore,

$$V(p) = 0.01 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Substituting 2.33α for β and solving gives us an alpha of 14.0 and a beta of 6.0. This model gives p_i values from about 0.5 to about 0.9, which seems like a practical interval.

Using this alpha and beta, twenty-five random deviates are drawn from the beta distribution as described above and used as multipliers of the dependent variable, Y, with the days in the order as defined by the simple multiple regression. At this point, there are twenty-five data points with different values for the resource variables from those originally collected. Rather than represent merely man hours available, they now are the estimates of man hours applied. A multiple rgression was run on this modified set of data. This regression yielded an F value of 1.33 and a correlation coefficient of 0.31.

Since twenty-five factorial different orderings of the days exist, it is unrealistic to examine each one. A simpler method is to interchange a pair of days which interchange results in a better model. The criterion used in this procedure to define the relative worth of a model is the F statistic. It is a reliable measurement of goodness of fit since it is the ratio of the regression mean square divided by the residual mean square. No direct comparison can be made between this F statistic and that of the simple multiple regression given in Section 2, since the degrees of freedom are now somewhat cloudy. However, the degrees of freedom are constant from this point on and the F statistic is a good relative measure between models in this procedure. Higher F values mean that the estimates of man hours applied are better fitted to the production data, intuitively have a greater likelihood of being true, and therefore yield a model with more accurate coefficients.

An examination of the residuals from the regression just run shows that the largest absolute value residual belongs to the first data point, which in the initial order is day 25. Its residual is -13.52, the negative sign indicating that the predicted value was greater than the observed. The observed value is, of course, not the actual value collected, but this term for the modified dependent variable will be used henceforth in this paper. The predicted value greater than the observed indicates that that day is less slack than the order indicates and should be moved further down the list. Since it is not known how much further, a simple interchange of the first two days is made, thereby moving the first day one position.

Keeping alpha and beta at the same values, a new regression is run with the different ordering. This yields an F of 1.93, showing that the interchange was an improvement on the original order. Again, examination of the residuals shows that the largest absolute value residual is 11.41 and belongs with the second day. It again has a negative sign, indicating that that day needs to be moved even further down the list. After interchanging it with the third day a new regression gives an F of 2.40. This continues in fact until day 25 has moved down into the thirteenth position with an F of 6.26. At this point the twenty-third data point, day 19, has the largest absolute value residual and is switched with the twenty-fourth data point giving an F of 6.54.

Although the interchange procedure could continue further, at this point it was decided that a better shape of the beta distribution might speed up the reordering process. A non-sophisticated technique was used consisting of simply running many regressions with the final order given in the previous paragraph, arbitrarily choosing a different alpha and beta for each one. From this sample the best regression was at alpha equal 30.0, beta equal 5.0 and F equal 45.54. The new mean of the beta distribution is 0.86 with a standard deviation of 0.06. This regression is significantly better than that which used an alpha of 14.0 and beta of 6.0. Using the new values, the reordering step continued as described above.

This process continues until the following point is reached. The order of the days is 16, 6, 21, 5, 24, 13, 25, 18, 4, 22, 11, 8, 9, 15, 2, 20, 3, 17, 7, 1, 14, 12, 10, 23, 19. Alpha is 25.00 and beta 5.25 yielding an F of 1041.04. The beta distribution has a mean of 0.83 and a standard deviation of 0.07. The coefficients are:

$b_0 = -0.5967$	$b_4 = 0.0154$
$b_1 = 0.0291$	$b_5 = 0.0682$
$b_2 = 0.0999$	$b_6 = 0.0897$
$b_3 = 0.1480$	

It is favorable to see that b_0 is very small since the beta distribution should be compensating for the slack. The residuals are also very small, the largest absolute value being 0.68.

It seems appropriate at this stage in the process to find the alpha and beta which maximize the F statistic for this order. The concern is that non-optimal alphas and betas can adversely affect the reordering process, and make further use of the above technique futile.

Figure 2 shows values of F at different values of alpha and beta for the above listed order of days. From visual inspection a mountain seems to exist with the ridge roughly following the line $\beta = 0.5\alpha - 7$. Figure 3 is an expansion of a segment of Figure 2, in which five slices have been taken across the mountain, where beta was kept constant at 5.25, 5.75, 6.25, 6.75 and 7.25 and alpha varied. It can be seen that the F values along each slice rise and fall, forming unimodal figures which are roughly parabolic. To locate the peak of each of the five parabolas, a modification of the Levenberg-Marquardt method for solving nonlinear least squares was employed using subroutine ZXSSQ of the IMSL package. Table 2 summarizes the results of this routine. These peaks represent the ridge of the mountain.

Regressions were run on each side of the peak points to assure that the ridge was located, and in only one case was there a slight discrepancy, which is reasonable since the curves may not have been perfectly parabolic. The regressions also give true, rather than estimated, F values at each of the five peaks. Table 3 gives the true optimal alphas and F values for the five curves.

Since the F values in Table 3 rise and fall, it is clear that the peak of the ridge fall within the range of the five slices. Because the ridge is undoubtedly unimodal, a search technique was employed to locate the true peak. The ridge line was defined as $\beta = 0.5444\alpha - 7.518$ from the five points in Table 3, and the initial search interval as

$$23.32 \leq \text{alpha}_{F(max)} \leq 25.31$$

At each step of the search, two points were picked off the line and their F values, F_i and F_j, were compared. By definition, if $F_i < F_j$, where $\text{alpha}_i < \text{alpha}_j$, the peak of the ridge must lie to the right of alpha_i. Therefore the segment of the line to the left of alpha_i can be eliminated from the search interval. The converse is, of course, also true. To maximize the reduction of the search interval at each step, alpha_i and alpha_j were chosen very close to the center of the interval and equidistant from it. This means that after each comparison, the search interval was reduced by slightly less than half. The finding of the peak was defined by a sufficiently small difference between F_i and F_j, the criterion used being that $|F_i - F_j| < 0.001$. By progressively limiting the area of search, an optimum was located at alpha equal 24.756, beta equal 5.959

and F equal 1694.57. Since the ridge line is not truly straight and therefore the equation above for it is only an approximation, alpha and beta combinations right around the maximum were checked by regression and it was determined that in fact the true optimum had been found as stated within the range alpha ± 0.05 and beta ± 0.05.

Using this alpha and beta, an exhaustive interchange of neighboring pairs was carried out which showed that days 1 and 14 should be reversed. Given this new order, the above procedure for locating $F_{(max)}$ was performed again. In this case, only four slices across the mountain were needed before the peak of the ridge was contained. The new optimum was located at alpha equal 24.31, beta equal 5.67 and F equal 2324.63.

Again, the interchange procedure yielded a new order with day 2 and 20 being reversed. Optimal F was found to be 2387.26, alpha 24.4 and beta 5.65. At this point no further interchanges improved the solution, so it was felt that the technique had located its final optimum. The coefficients for the regression are:

$b_0 = -2.304$	$b_3 = 0.150$
$b_1 = 0.028$	$b_4 = 0.015$
$b_2 = 0.103$	$b_5 = 0.069$

$b_6 = 0.088$

The mean of the final form of the beta distribution is 0.81 with a standard deviation of 0.07 and peak at 0.83. The largest absolute value residual is 0.5. The final order of days is 16, 6, 21, 5, 24, 13, 25, 18, 4, 22, 11, 8, 9, 15, 20, 2, 3, 17, 7, 14, 1, 2, 10, 23, 19.

The negative intercept in the final equation is not disturbing as it is for Martin's results because in this case it does not violate the assumption that the dependent variable represents an upper limit on the hours worked. The intercept in this procedure quantifies the daily average idle time which the beta distribution could not account for. Since its sign is negative it actually represents the average additional busy time per day. However, it is sufficiently small that its addition does not cause the assumption on the original resource variable to be violated.

Figure 4 is a normal probability plot for the residuals from the final form of the regression. Note that the residuals are all very small, the largest absolute value residual being 0.50 as opposed to 10.21 for Richardson's results. Very small residuals are important because the p_i term in the regression should be compensating for the daily fluctuation of work activity. The W test, run on the residuals to test for normality, yielded a Z value of -0.0594 and from a normal distribution table, $Pr(Z \leq -0.0594) = 0.4761$. This probability is very close to that arrived at by the multiple regression in Section 2. There are other regressions with different alpha/beta pairs within this order in which the hypothesis of normality could be accepted with more certainty, however a trend is not discernible. It is unclear as to whether no pattern exists at all or no pattern exists within this order. If a different criterion for optimality is used, in this case optimizing the normality of residuals, the interchange of days procedure might yield a different reordering pattern in an attempt to maximize the probability that the residuals are normally distributed.

It must be stated that the original matrix is somewhat ill-conditioned. That is, the regression coefficients tend to be unstable given change in the dependent variable. That fact does not affect the validity of the procedure outlined in this paper, just the trustworthiness of the results which are gotten from this matrix. In order to test the unstableness of the coefficients, it was decided to run the procedure over a subset of the data, in this case the first fifteen days. The hope was that the coefficients would not alter drastically. At the point and order of optimality the results are:

$b_0 = -2.681$	$b_4 = 0.016$
$b_1 = 0.032$	$b_5 = 0.077$
$b_2 = 0.11$	$b_6 = 0.091$
$b_3 = 0.168$	

for the fifteen days. Alpha equals 16.355 and beta equals 1.92. Although the order of the days in the subset remained the same as they were in the full data set, there is somewhat of a change in the final beta distribution. Here the mean is 0.89 and standard deviation of 0.07, which means that on the average, eight percent more of each day was applied directly to production, not including the effect on the intercept, which is fairly small. It is also interesting to note that it took longer to turn out all of the six products in the subset than in the complete set of observations. This phenomenon is likely to occur since busy days essentially define the coefficients and as you add days, busy days are likely to be included, thereby driving the values of the coefficients down. Inversely, when you take away days, the coefficients will tend to rise. Days 19, 23, and 17, three of the ten busiest days, were excluded from the subset and therefore the coefficients rose when only the first 15 days were run. The coefficients which changed the most were for products 3 and 5 and days 19 and 23 represent among the busiest for these products. The important fact however, is that the coefficients did not vary widely from the results of the full twenty-five days. The change in the resource variables, given the amount of change in the form of the beta distribution, did not give drastically different results in the work standard figures. Therefore, in this case, it does not appear that the matrix is sufficiently ill-conditioned as to make results questionable.

A final caution must be given about the results obtained by this procedure relative to some further experimental work performed on the data. The worth of the results is predicated on the assumptions that 1) simple interchanges of days will eventually sort the data into the best order, 2) the reordering is not overly sensitive to the values of alpha and beta being used, 3) within any order the surface of the mountain, which is described by F values, will be regular, i.e. unimodal in all directions, and 4) the

F statistic is an appropriate measure of performance for this procedure. This last assumption is, in fact, the most important one, since the criterion for optimality can effect all the previous assumptions. Since testing all of these assumptions would yield an enormous set of possible experiments, it was decided to continue to rely on assumptions one and four, i.e. the simple interchange of days will eventually optimize the order and the F statistic is a stable measure in this case.

To test the second assumption, the same procedure was rerun, omitting the finding of better alpha/beta pairs after every several interchanges. Another small change was introduced. Rather than base the decision for interchange on the largest absolute value residual, the interchange of all neighboring pairs was run and the one which produced the largest F value was the one decided upon. This latter change makes the procedure more consistant with the fourth assumption, which is that the F statistic is a good criterion for the model. The identical interchanges were made in any case until the point where the two procedures diverged, which was when the first procedure changed alpha/beta values.

Eventually this technique reached the point where no additional interchanges improved the F statistic. This order differed from the final order derived previously. The F statistic was lower than the optimal given, which is not surprising since the form of the beta distribution had not been optimized in this case. Since it was not clear if an absolute final order had been found or only the best order for an alpha of 14.0 and a beta of 6.0, the same search procedure for finding the peak of the mountain as outlined before was used to locate the best alpha/beta pair. After this was done, interchanges again began to improve the order using the new alpha and beta. At every point when reordering stopped, the best alpha/beta values were located relative to the optimizing criterion and reordering resumed. Finally the point was reached where reordering and searching converged on the optimal.

Unfortunately this second optimal solution was very different than that given for the first optimal. Relative to the F statistic, it was also better. It is unfortunate because it shows that either the second assumption can not be relied upon and that the reordering is sensitive to alpha and beta, or that the F statistic is not a stable criterion for optimality. Here F equals 2909.46, alpha equals 14.74, beta equals 9.77, E(p) equals 0.60 and V(p) equals 0.009. The coefficient are:

$b_0 = -6.5909$ $\qquad$ $b_4 = -0.004$

$b_1 = 0.030$ $\qquad$ $b_5 = 0.0376$

$b_2 = 0.140$ $\qquad$ $b_6 = 0.223$

$b_3 = 0.092$

The order is 6, 15, 24, 9, 21, 16, 18, 5, 17, 4, 22, 14, 7, 1, 20, 23, 12, 10, 11, 13, 2, 8, 3, 19, and 25. This solution is disappointing for several reasons. Most disconcerting is the negative coefficient for product four. It obviously can not be accepted as a valid work standard figure. Another problem is the degree of change in the coefficients. A simple test of hypothesis stating that the b_i's, for i = 1 to 6, are the same for these results versus the other results must be rejected for each at the 95 percent confidence level using a two-tailed t test. The relative position of the days, along with the moments of the beta distribution and the magnitude of b_0, all of which of course affect the dependent variable, can have a major impact on the coefficients. For example, day 25, which essentially dominates the coefficient for product six, moved from the seventh to the twentyfifth position. The impact of this is that the dependent variable, the applied man hours, has a larger value than had it remained in the seventh position. Nevertheless, since the beta distribution is shifted to the left, the modified dependent variable is essentially the same as it was in the seventh position with the other form of the distribution. However, the difference in the b_0's adds an extra 4.7 hours onto the applied man hours, thereby allowing b_6 to become larger. The fact that the change in the coefficients can be explained does not lessen the problem that the assumptions seem in doubt.

Ignoring the implications of this for the moment, and assuming that the reordering is sensitive to the alpha/beta pair being used, it stands to reason that in all probability the optimal solution has not been located in either of the two procedures. To test this, the procedure was run again identically to the last time, the only change being that after each interchange the search procedure was used to find the peak of the mountain for that order. Problems developed here with the search procedure. Starting with the order from the multiple regression, it was discovered that the surface of the mountain was not unimodal, but very jagged instead. This fact belies the third assumption, at least in situations where the order being used for the optimization is very bad. It also made the defined search procedure impossible to use since it depends upon that assumption.

What this all seems to point to is that the F statistic may not be the most stable criterion for this procedure. In the first place, it can yield an irregular surface for some orders. Secondly, the movement toward optimality is discontinuous, at least with regard to the interchange procedure being used. In other words, it was envisioned that one hugh mountain existed with a series of smaller peaks on it, one for each of the twenty-five factorial different orderings. It was also thought that interchanges would move you from peak to peak until the ultimate summit had been reached. By virtue of the fact that two peaks were found from which further movement was impossible implies that either there is more than one large mountain or that the simple interchange of days is not adequately strong to allow movement at all points. This remains an open question.

SECTION 4

CONCLUSIONS

Although it has been shown that this particular technique relies on some dubious assumptions on a detailed level, the use of this type of procedure is still valid. As long as the dependent variable is weak and merely represents a boundary condition, no procedure which uses it at face value is valid. Since the multiple regression and linear programming techniques do just this, they are improperly used in this case. Presuming a linear relationship exists, one is on much stronger analytical ground by massaging the weak dependent variable than by leaving it alone.

This procedure may not have located the abolute optimal, which would be very difficult to prove in any case, but did give solutions in which the residuals were very small and the assumption on the resource variable was not violated. Both of these are very positive results. There may ultimately be better models in which something other than the beta distribution is used and/or in which a non-linear relationship is found for these data. However, for a different set of data, the new and more complex model may not be as valid. Therefore it is practical to use the form of the model as shown in this paper which assumes the simplest relationship among the variables and uses a very flexible distribution in the unit interval. The chance therefore that this model will be suitable for other data sets is increased.

An interesting sidelight in the final results is the tendency for Mondays to be among the slackest days. In fact, the three slackest days in the first optimal given are all Mondays. The probability that this occurred randomly is 0.004. Tuesdays, on the other hand, tend to be among the busiest days, especially in the first optimal, with four of the five in the busier half of the ordering of days. This could occur for several reasons. There could be a pattern to the company's sales orders due to business practice, the mail, or whatever. More likely, it could have occurred because people tend to work more slowly on Monday and try to catch up the following day. The fact that this phenomenon, generally accepted as being true, was shown by this procedure, gives the procedure extra credibility.

Future Work

There are some areas within this study where improvement might be made or the procedure made more efficient. The most important would be to determine if there is a more appropriate and stable criterion for the model's usefulness than the F statistic. Examples would include optimizing the normality of the residuals or minimizing the absolute value of the intercept. It is also possible that more efficient and assured reordering techniques exist. Perhaps the simple interchange of days is too subtle for the technique.

Secondarily, more mathematically sophisticated methods of locating optimal alphas and betas could be derived. Likewise, the process of flip-flopping between reordering and searching might be expedited, if it were determined how sensitive reordering is to alpha and beta. A different starting order may be recommended. Something other than the beta distribution may be used. Non-linear models might be tried. The intercept, representing average idle time, should approach zero, since the beta distribution compensates for idle time, and perhaps the iterations would go faster if b_0 were forced to zero throughout the regressions.

Ultimately, the verification of this type of approach should be carried out by performing simultaneous studies. In the one, traditional industrial engineering methods of work measurement should be used. Concurrently, a model using the philosophy of the approach developed in this paper should be derived from production data over the same time period. The results should be compared.

SECTION 5

SUMMARY

Work standards are not always readily obtainable by traditional industrial engineering methods, especially when indirect labor is involved. Likewise if production data with a weak dependent variable are being used, traditional mathematical methods such as linear programming and simple multiple regression are not appropriate. The approach proposed in this paper is to modify the dependent variable, Y_j, using random deviates which approximate order statistics from the beta distribution so that the new dependent variable, Y_j', more realistically represents the time actually applied to producing units of work. Multiple regressions are then performed.

Handled are the two simultaneous problems of ordering the data such that they match the random deviates from the beta distribution and of locating alpha and beta such that the F statistic from the analysis of variance on the regression is maximized.

Date	Day No.	Y	X_1	X_2	X_3	X_4	X_5	X_6
3/30	1	80	197	155	211	360	171	17
3/31	2	80	187	113	194	236	196	71
4/1	3	80	394	125	204	113	128	53
4/2	4	80	187	61	191	407	181	70
4/3	5	80	121	25	193	317	210	86
4/6	6	72	136	67	140	329	210	18
4/7	7	64	187	24	169	402	174	69
4/8	8	80	133	122	216	259	104	84
4/9	9	80	172	97	273	329	100	0
4/10	10	72	214	55	231	406	114	69
4/13	11	78	383	100	178	396	67	69
4/14	12	80	325	115	212	381	162	36
4/15	13	80	321	100	152	82	195	74
4/16	14	80	216	30	203	335	299	67
4/17	15	80	96	32	282	514	130	34
4/20	16	80	221	25	204	96	133	68
4/21	17	80	218	50	221	404	225	47
4/22	18	80	170	94	151	621	188	58
4/23	19	80	247	83	282	175	124	76
4/24	20	80	266	52	199	396	203	74
4/27	21	80	279	75	192	419	76	47
4/28	22	74	160	17	160	263	281	85
4/29	23	80	315	49	219	122	324	40
4/30	24	80	245	91	199	546	89	24
5/1	25	94	330	116	184	86	192	128

Table 1

The Production Data

	Alpha at Peak	F at Peak	Corresponding Beta
Curve A	27.09	1613.25	7.25
Curve B	26.26	1656.26	6.75
Curve C	25.31	1688.01	6.25
Curve D	24.34	1685.91	5.75
Curve E	23.31	1647.79	5.25

Table 2

Alpha and F Values at the Peaks of the Curves as Found by the Modified Levenberg-Marquardt Algorithm

	Alpha at Peak	F at Peak	Corresponding Beta
Curve A	27.13	1613.95	7.25
Curve B	26.26	1659.45	6.75
Curve C	25.31	1688.94	6.25
Curve D	24.34	1691.75	5.75
Curve E	23.31	1655.37	5.25

Table 3

True Alpha and F Values at the Peaks of the Curves

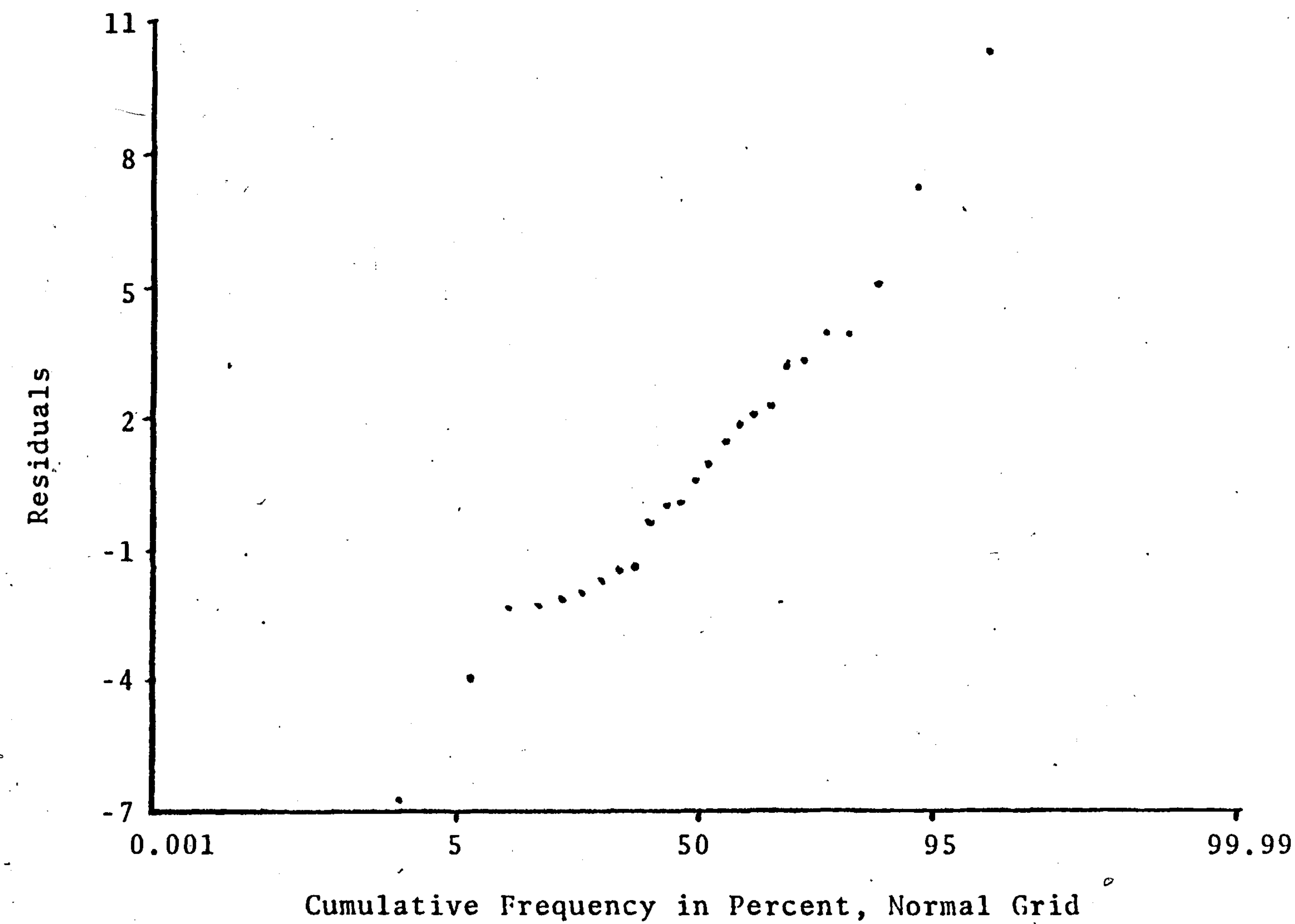

Figure 1 - Normal Probability Plot of Residuals from the Simple Multiple Regression

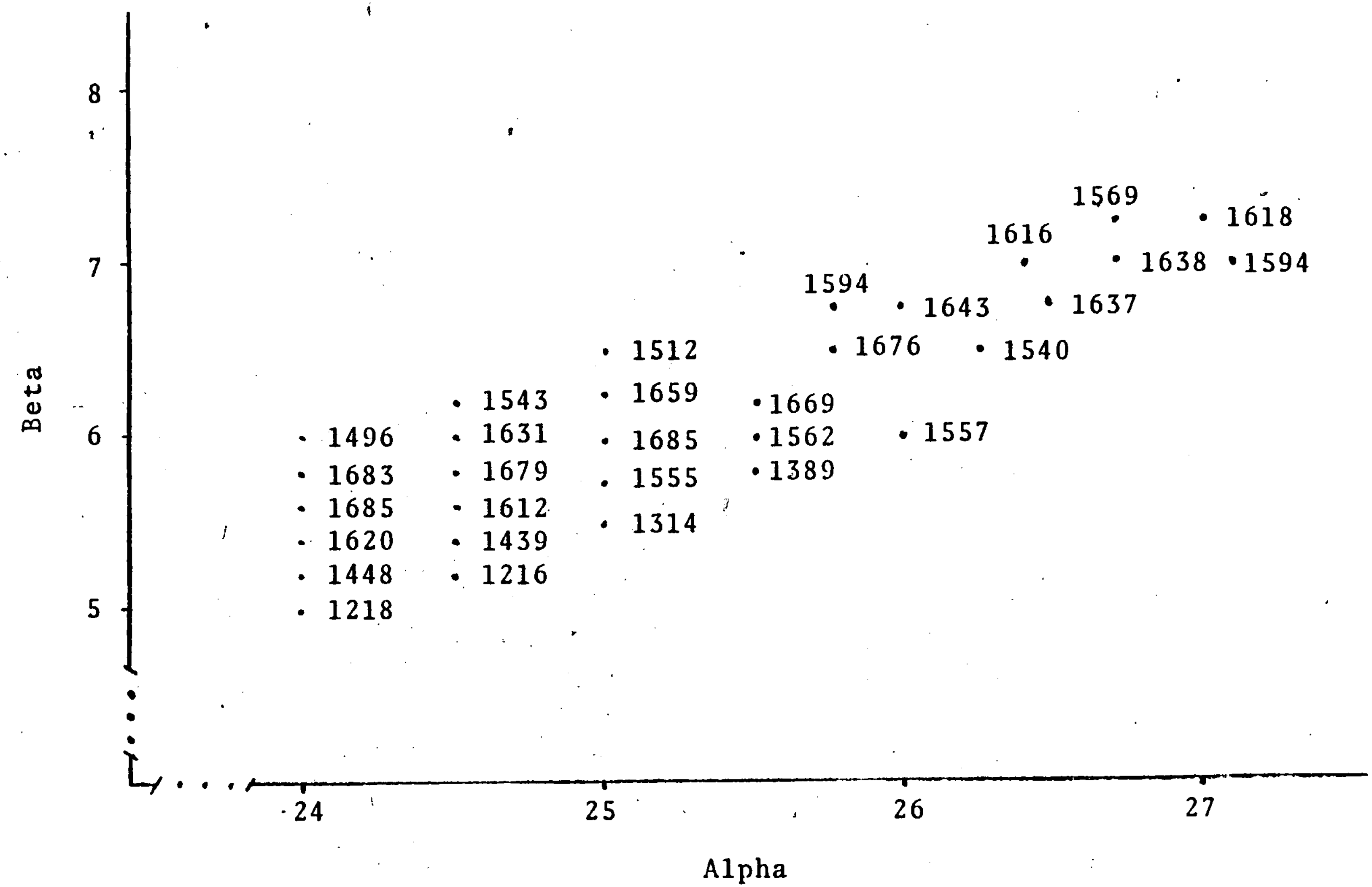

Figure 2 - F Values from the Regression at Varying Levels of Alpha and Beta

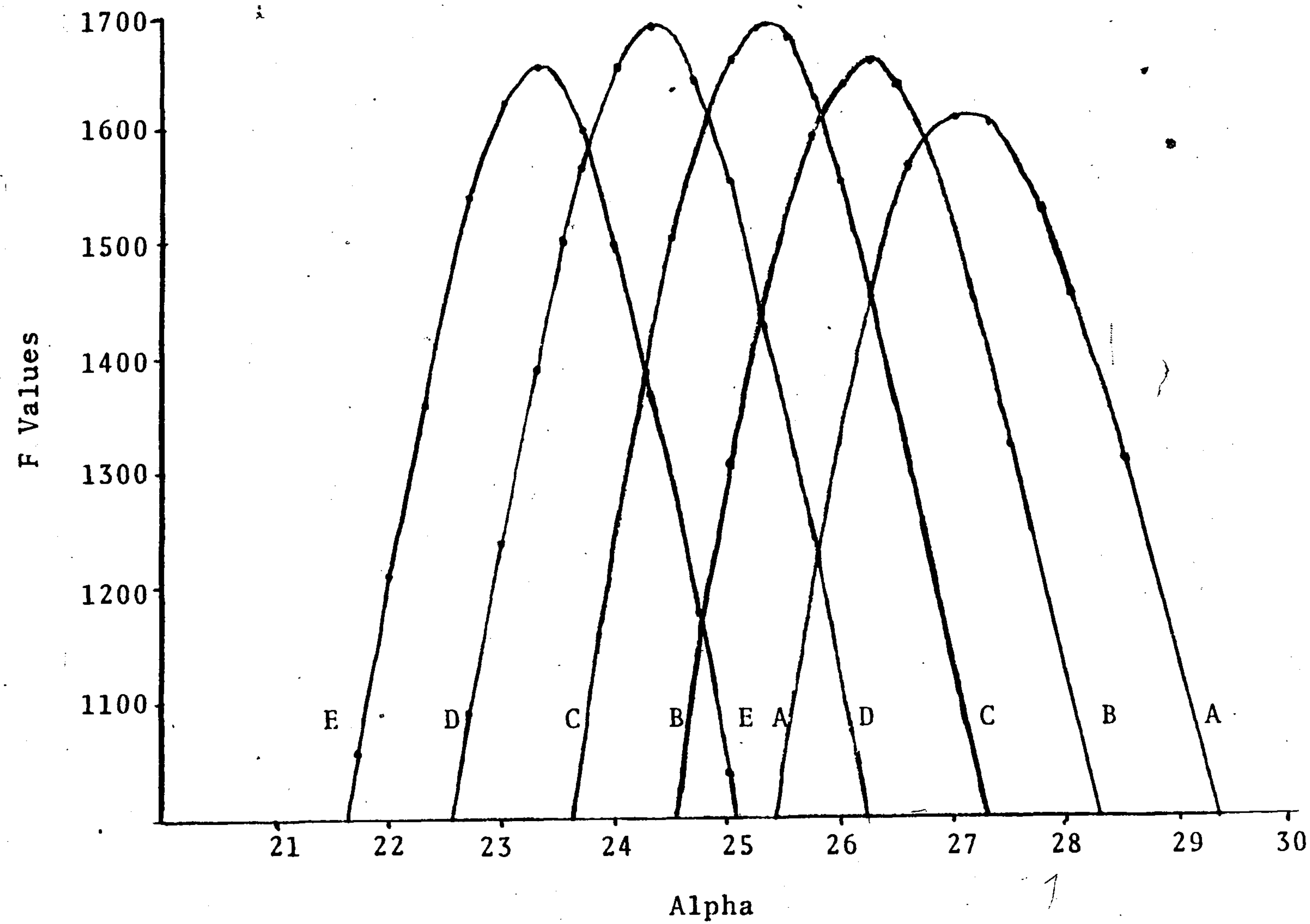

Figure 3 - Alpha Versus F Curves at Five Levels of Beta

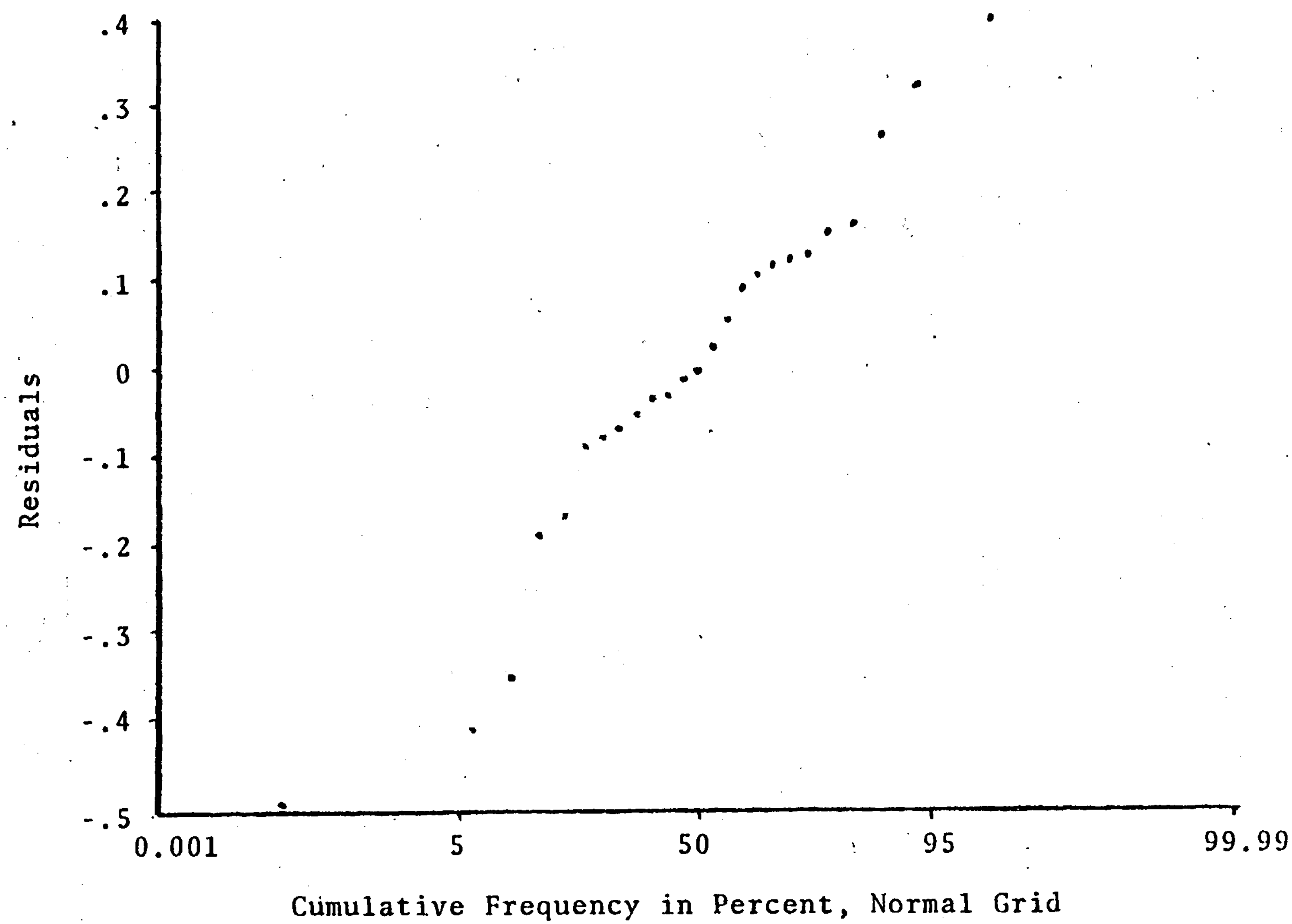

Figure 4 - Normal Probability Plot of Residuals After the Optimization Procedure

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VITA

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